In the following the field of vector spaces and algebras is taken to be the field of complex numbers and σ denotes spectrum.

1. *Obtain the spectral decomposition for following matrices, that is, write them as unitary conjugates of diagonal matrices and also write them as linear combination of projections.*

$$M = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}, \; N = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

Solution: Let us assume that the matrix representation of M corresponds to the basis $\{e_1, e_2\}$. With respect to this basis, the basis vectors are represented as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Now the given matrix M is self-adjoint and its eigenvalues will correspond to the roots of the equation

$$(5 - \lambda)^2 - 4 = 0$$

It is easy to see that the roots are $3, 7$. Now corresponding to these eigenvalues, let us find out the respective eigen-vectors. If the eigen-vector for the eigenvalue 3 is written as $\begin{pmatrix} x \\ y \end{pmatrix}$ then we need to solve for the linear equations

$$\begin{aligned} 5x + 2y &= 3x \\ 2x + 5y &= 3y. \end{aligned}$$

to get a representation of the eigen-vector with respect to the basis $\{e_1, e_2\}$ upto the constant $\lambda \in \mathbb{C}$. This then gives us the solution $x = -y$. Consider for example $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as an eigen-vector. Similarly a typical element of the eigenspace for the eigen-value 7 would be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The corresponding normalized eigen-vectors are going to be $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$Now we would like to solve for the basis change matrix from $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ to $\{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}\}$[1]. If we write down this basis change matrix as $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$, we get that for the *given representations* the basis change matrix is $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$. Correspondingly the inverse matrix that would be the basis change matrix taking $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and

[1] It is easy to see that this forms an orthonormal basis, directly from the fact that they are the normalized eigen-vectors corresponding to distinct eigen-values.

$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$ **to** $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ **would be given as** $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ **Thus the representation of** M **as a unitary conjugate of a diagonal matrix would be** $\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$.

Next we would like to break down M **as a linear combination of projection matrices. Consider the eigenspaces of** M **corresponding to the eigen-vectors** $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. **These are the two one-dimensional invariant subspaces for the matrix** M. **We thereby see that** M **can be written as** $M = 7P_1 + 3P_2$, **where** P_1 **is the projection from** $\mathbb{C}^2$ **to the eigenspace of eigenvalue** 7 **and similarly** P_2 **is the projection onto the other eigenspace. Clearly if we choose the eigen-vectors of** M **in the above order as a basis for** $\mathbb{C}^2$, P_1 **shall look like** $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, **while** P_2 **shall be** $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. **Next we should conjugate the matrices above by the basis change matrices to arrive at the required projections with respect to the original basis for which** M **has the given particular representation.**[2] **This then gives us that**

$$P_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Similarly we have the following representation for P_2 **corresponding to the other eigenvalue**

$$P_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$$

Therefore we have the following representation of M **as a linear combination of projections, namely**

$$\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} = 7 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$$

Next we do the same analysis for the other matrix N. **Let us assume that the given matrix representation of** N **is with respect to the basis** $\{e_1, e_2, e_3\}$ **in** $\mathbb{C}^3$. **First we find out the eigenvalues of** N **which are** $0, 4$ **with the eigenspace of** 4 **having dimension** 2. **Solving for the normalized eigenvectors, we get the following orthonormal basis for the eigenspace of** 4, **namely,** $\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \}$ **and the one-dimensional eigenspace of** 0 **is given by** $\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$.

Now for this given orthonormal basis the matrix corresponding to the change of basis from $\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$ **to the above one** $\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \}$ **is given by**

[2] Note that the representation of the projection P_1 is independent of the choice of the eigen-vector from the eigenspace of the given eigenvalue.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ **which is a unitary matrix with the same matrix as its adjoint. Thus the decomposition of N as a unitary conjugate of a diagonal matrix shall be given as**

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Next we look at the decomposition of N as a linear combination of projections $N = 4P_1 + 0P_2$, since $4, 0$ are the two eigenvalues of N. Therefore $N = 4P_1$ which in the matrix representation becomes

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} = 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

□

2. *Let X, Y be Banach spaces. Show that $K(X,Y)$, the space of all compact operators from X to Y is a closed subspace of the space $B(X,Y)$ of all bounded operators from X to Y.*

Solution: If $\{T_n\}_n$ is a Cauchy sequence of compact operators in $K(X,Y)$ converging to the operator T in $B(X,Y)$, it is enough to show that T is a compact operator. For Banach spaces, we know that an operator T is compact if the image of the unit ball under T has compact closure in the norm topology of the Banach space in the co-domain. Now given $\epsilon > 0$, there exists $N \geq 1$ such that $\|T - T_N\| \leq \frac{\epsilon}{2}$. Now if S is the unit ball of X, T_N being a compact operator one can cover $T_N(S)$ by finitely many open balls of radius $\frac{\epsilon}{2}$. From $\|T - T_N\| \leq \frac{\epsilon}{2}$, keeping the same centres for the balls and increasing the radius of the balls by $\frac{\epsilon}{2}$, one can cover the set $T(S)$. Hence T is a compact operator. □

3. *Let $\mathcal{F}$ be the algebra of all matrices of the form:*

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

with $a, b \in \mathbb{C}$. Show that $\mathcal{F}$ with

$$\| \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \| = |a| + |b|$$

is a commutative Banach algebra (You must verify all the axioms). Compute the spectrum of this Banach algebra.

Solution: Clearly $\mathcal{F}$ is a subspace in $M_2(\mathbb{C})$ and is also closed and commutative under the usual matrix multiplication, namely

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} ac & ad + bc \\ 0 & ac \end{bmatrix}$$

. We next show that it is closed with respect to the norm given above. Namely if $\{\begin{bmatrix} a_n & b_n \\ 0 & a_n \end{bmatrix}\}_n$ is a Cauchy sequence of matrices in the given norm, we have that $\{|a_n| + |b_n|\}_n$ is a Cauchy sequence in $\mathbb{C}$. This would imply that both $\{a_n\}_n$, $\{b_n\}_n$ are Cauchy sequences in $\mathbb{C}$ and hence converges to some a, b respectively in the complete space $\mathbb{C}$. Thus we get that the above sequence of matrices shall converge to $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ and hence $\mathcal{F}$ is a closed subspace of the Banach space $M_2(\mathbb{C})$ and is therefore a commutative Banach algebra.

We next try to compute the spectrum of $\mathcal{F}$, that is the collection of all multiplicative linear functionals on $\mathcal{F}$. Now $\mathcal{F}$ is a 2-dimensional subspace of $M_2(\mathbb{C})$, spanned by the following basis

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hence given a multiplicative linear functional $\phi\colon \mathcal{F} \to \mathbb{C}$, its enough to deduce its value on the above two matrices A, B. But $B^2 = 0$ and hence

$$\phi(B^2) = (\phi(B))^2 = 0.$$

This gives that $\phi(B) = 0$ for all multiplicative linear functionals on $\mathcal{F}$. Next, we see that $A^2 = A$, which implies $\phi(A)^2 = \phi(A)$ and therefore $\phi(A)[\phi(A) - 1] = 0$. This gives that $\phi(A) = 0$ or 1. Therefore the set of multiplicative linear functionals on $\mathcal{F}$ consists of two elements, the zero functional and the other taking the value 1 on A and 0 on B. □

4. *Let A be a unital Banach algebra. Consider a, b in A.*

   (a) *Show that if $(1-ba)$ is invertible then so is $(1-ab)$. (Hint: If $c = (1-ba)^{-1}$, then $(1-ab)^{-1} = 1 + acb$).*

   (b) *Show that $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.*

Solution:

(a) Given $(1 - ba)$ is invertible, the inverse is given as a convergent power series

$$\tfrac{1}{(1-ba)} = 1 + ba + (ba)^2 + (ba)^3 + \ldots = c$$

Similarly the inverse of $(1 - ab)$ will be given by the power series

$$\tfrac{1}{(1-ab)} = 1 + ab + (ab)^2 + (ab)^3 + \ldots$$

It is then clear to see that

$$\tfrac{1}{(1-ab)} = 1 + acb$$

Given that $(1 - ba)$ is invertible which in turn would mean that c is a convergent power series, we have that $1 + acb$ is a convergent power series and hence $(1 - ab)$ is also invertible.

(b) **From the above proof, note first that the converse also holds. Namely, if $(1 - ab)$ is invertible then $(1 - ba)$ should also be invertible. Now given $\lambda \neq 0$, we have $(\lambda - ab) = \lambda(1 - \frac{ab}{\lambda})$. Therefore from part (a), $(\lambda - ab)$ is invertible if and only if $(\lambda - ba)$ is invertible. Hence except for 0, we see that $\sigma(ab) \setminus 0 = \sigma(ba) \setminus 0$. Thus $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.**

$\square$

5. *Let $\mathcal{E} = C[0,1]$ be the Banach algebra of complex valued continuous functions on $[0,1]$. Let I be the ideal, $I = \{f : f \in \mathcal{E}, f(0) = f(1) = 0\}$. Show that the quotient space $\mathcal{E}/I$ is isomorphic to $\mathbb{C}^2$.*

   Solution: Consider the map $\Phi \colon \mathcal{E} \to \mathbb{C}^2$, given as $f \mapsto (f(0), f(1))$. Clearly this a linear map that also preserves the multiplicative structure. It is easy to see that the ideal I is the kernel of this map and the map is surjective. To see surjectivity of Φ, consider any element $(a, b) \in \mathbb{C}^2$. Then the function $g \colon [0,1] \to \mathbb{C}$ given as $g(t) = a(1-t) + bt$ is a continuous function such that $g(0) = a$ and $g(1) = b$. Hence by the first isomorphism theorem, we have that the quotient space $\mathcal{E}/I \cong \mathbb{C}^2$. $\square$

6. *Let $\mathcal{A}$ be a unital commutative Banach algebra. Define the Gelfand map for $\mathcal{A}$ and show that the Gelfand map is a contractive homomorphism*

   Solution: Let $\mathcal{M}(\mathcal{A})$ be the maximal ideal space for $\mathcal{A}$, that is the collection of all multiplicative linear functionals on A. Then the Gelfand map is defined as $\Phi \colon \mathcal{A} \to C(\mathcal{M}(\mathcal{A}))$, which is given as $\Phi(a)(\psi) = \psi(a)$, for all $\psi \in \mathcal{M}(\mathcal{A})$. To see that it is a homomorphism, consider the product of any two elements $a, b \in \mathcal{A}$, then $\Phi(ab)(\psi) = \psi(ab) = \psi(a)\psi(b)$, for all multiplicative linear functionals ψ in $\mathcal{M}(\mathcal{A})$. Therefore $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in \mathcal{M}(\mathcal{A})$ and hence the Gelfand map is a homomorphism. Now one needs to show that Φ is a contraction, that is, $\|\Phi(a)\| \leq \|a\|, \forall a \in \mathcal{A}$. Now $\|\Phi(a)\| = \sup_{\psi \in \mathcal{M}(\mathcal{A})} |\psi(a)|$. But the set $\{\psi(a) : \forall \psi \in \mathcal{M}(\mathcal{A})\}$ is contained in the spectrum $\sigma(a)$ of the element a. Hence $\|\Phi(a)\| \leq \boldsymbol{spr}(a)$, where $\boldsymbol{spr}(a)$ is the spectral radius of a. And we know that $\boldsymbol{spr}(a) \leq \|a\|$ for all a in $\mathcal{A}$. Hence $\|\Phi(a)\| \leq \|a\|$ for all a in $\mathcal{A}$. Therefore the Gelfand map Φ is a contractive homomorphism. $\square$

7. *Consider the set up of question 4. Suppose $\lambda \cdot 1 = ab - ba$, where λ is a scalar, show that $\lambda = 0$. (Hint: If $\lambda \neq 0$, arrive at a contradiction with 4(b) by considering $\sigma(ab)$ and $\sigma(ba)$.)*

   Solution: Suppose $\lambda \neq 0$. This would mean that $ba = ab - \lambda \cdot 1$. But then we have that $\sigma(ba) = \sigma(ab) + \lambda$, where the addition operation is with respect to that in $\mathbb{C}$. But this contradicts the statement of question 4(b) that $\sigma(ab) = \sigma(ba)$, hence $\lambda = 0$.